4. Advanced Solution Methods

4.1 Transformation of Cauchy-Euler to Constant Coefficient ODEs

Transformation Theorem: The Cauchy-Euler equation $ax^2 y'' + bx y' + cy = f(x)$ can be transformed into an ODE with constant coefficients using a logarithmic change of variable.

> **Substitution Variable:** $t = \ln x \Rightarrow x = e^t$
> **First Derivative:** $x y' = dy/dt = \dot{h}$
> **Second Derivative:** $x^2 y'' = d^2y/dt^2 - dy/dt = \ddot{y} - \dot{h}$
> **Transformed ODE:** $a\, d^2y/dt^2 + (b - a)\, dy/dt + cy = f(e^t)$

4.2 Variation of Parameters for Second-Order ODEs

Method Theorem: For a non-homogeneous equation in standard form: $y'' + P(x)y' + Q(x)y = g(x)$, with known complementary solutions y_1, y_2:

> **Particular Solution:** $y_p(x) = -y_1 \int [\, y_2\, g(x) / W \,]\, dx + y_2 \int [\, y_1\, g(x) / W \,]\, dx$
> **Wronskian:** $W = y_1\, y_2' - y_2\, y_1'$

4.3 Series Solutions of ODEs (about Ordinary Points)

- **Ordinary Point Definition:** A point $x = x_0$ is an ordinary point of $y'' + P(x)y' + Q(x)y = 0$ if both $P(x)$ and $Q(x)$ are analytic at x_0.
- **Power Series Trial Form:** The general solution can be written as:

> **Series Form:** $y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$

- **Existence Theorem for Series Solutions:** If x_0 is an ordinary point, then every solution is analytic at x_0. The power series converges at least for $|x - x_0| < R$, where R is the distance to the nearest singular point of the ODE in the complex plane.
- **Recurrence Relations:** Substituting the series and equating coefficients of each power of x to zero yields relations. For example:

> **ODE:** $y'' - y = 0 \Rightarrow$ Recurrence: $a_{n+2} = a_n / [(n+2)(n+1)]$
> **ODE:** $y'' + y = 0 \Rightarrow$ Recurrence: $a_{n+2} = -a_n / [(n+2)(n+1)]$

4.4 The Frobenius Method (about Regular Singular Points)

- **Regular Singular Point Definition:** A point $x = x_0$ is a regular singular point of $y'' + P(x)y' + Q(x)y = 0$ if it is not ordinary, but $(x - x_0)P(x)$ and $(x - x_0)^2Q(x)$ are both analytic at x_0.
- **Frobenius Theorem:** Let $x = 0$ be a regular singular point of the ODE. Then there exists at least one solution of the form:

> **Trial Solution:** $y(x) = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r}$ (with $a_0 \neq 0$)

where the exponent r is determined by the indicial equation obtained from the lowest power of x.